MTH 166

Lecture-35

Revision of Unit-6 and MCQ Practice

Line Integral w.r.t an Arc Length

Let the parametric representation of curve C be:

$C: x = x(t), y = y(t), z = z(t); \quad a \le t \le b.$

Let $f(x, y, z)$ be a scalar field function.

Then line integral of scalar function f over curve C w.r.t arc length s is given by:

$$I = \int_C f(x, y, z) ds$$

$$= \int_{t=a}^{t=b} \left[f(x(t), y(t), z(t)) \frac{ds}{dt} \right] dt$$

$$= \int_{t=a}^{t=b} \left[f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \right] dt$$

Line Integral of a Vector Field (Work Done)

Let us consider a vector field $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

We write $\overrightarrow{v(t)} = v_1(t)\hat{\imath} + v_2(t)\hat{\jmath} + v_3(t)\hat{k}$; $\quad a \leq t \leq b$

Let curve C is represented by: $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

We write C: $\overrightarrow{r(t)} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

Then line integral of vector $\vec{v}$ over curve C given by:

$$I = \int_C \vec{v}.\overrightarrow{dr} = \int_C \left[\overrightarrow{v(t)}.\frac{\overrightarrow{dr}}{dt}\right] dt$$

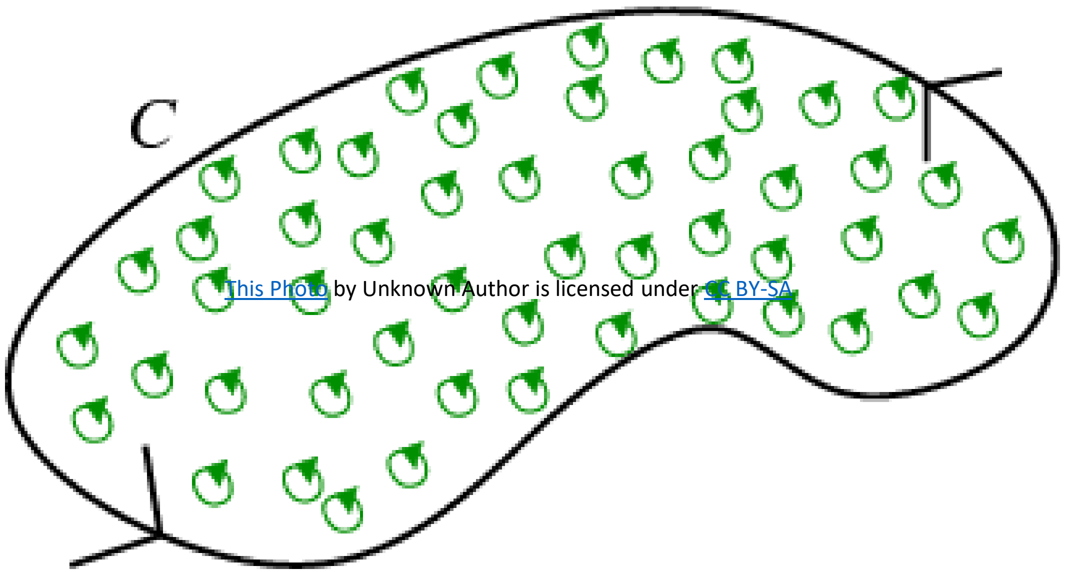
$$= \int_{t=a}^{t=b} \left[\left(v_1(t)\hat{\imath} + v_2(t)\hat{\jmath} + v_3(t)\hat{k}\right).\left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right)\right] dt$$

$$= \int_{t=a}^{t=b} \left[\left(v_1(t)\frac{dx}{dt} + v_2(t)\frac{dy}{dt} + v_3(t)\frac{dz}{dt}\right)\right] dt$$ This also called as **Work Done.**

Greens’ Theorem:

Let C be a piecewise smooth simple closed curve bounding a region R traced in anticlockwise direction. If f and g are two scalar functions which are continuous and have continuous first order partial derivatives on R, then:

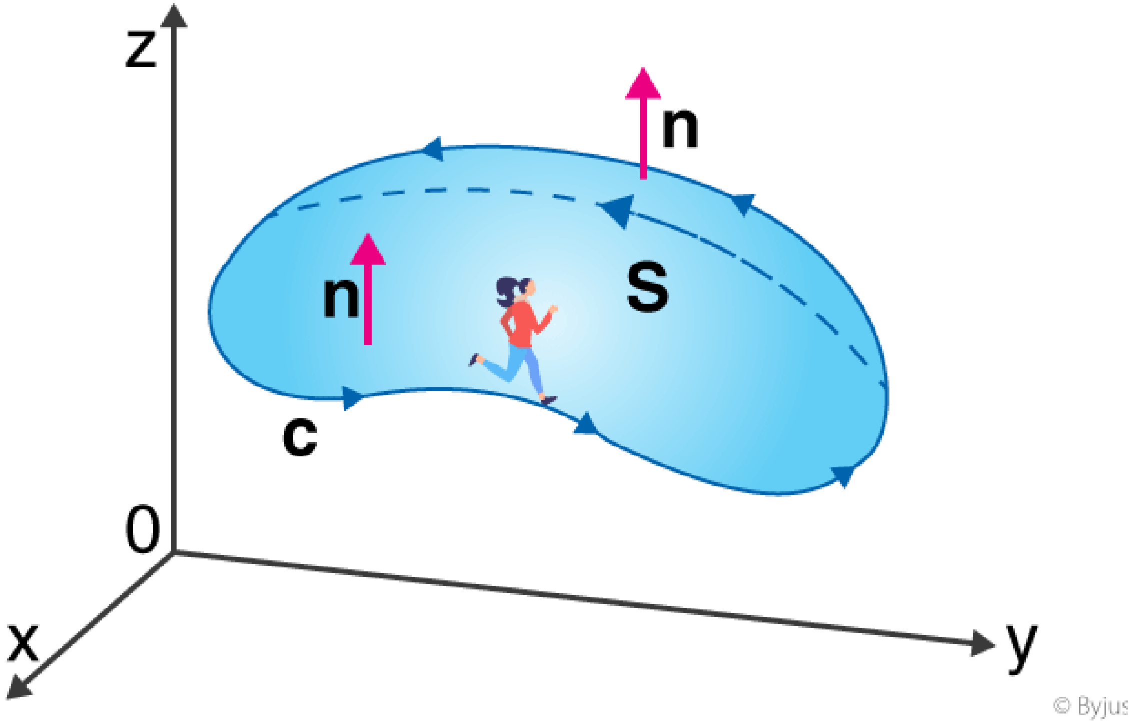
$$\oint_C f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$



<u>Stoke's Theorem</u>

Let S be a piecewise smooth orientable surface bounded by a piecewise smooth simple closed curve c traced in a positive direction. Let V be a vector which is continuous and have continuous first order partial derivatives. Let n is a unit normal vector drawn outward to the surface S. Then:

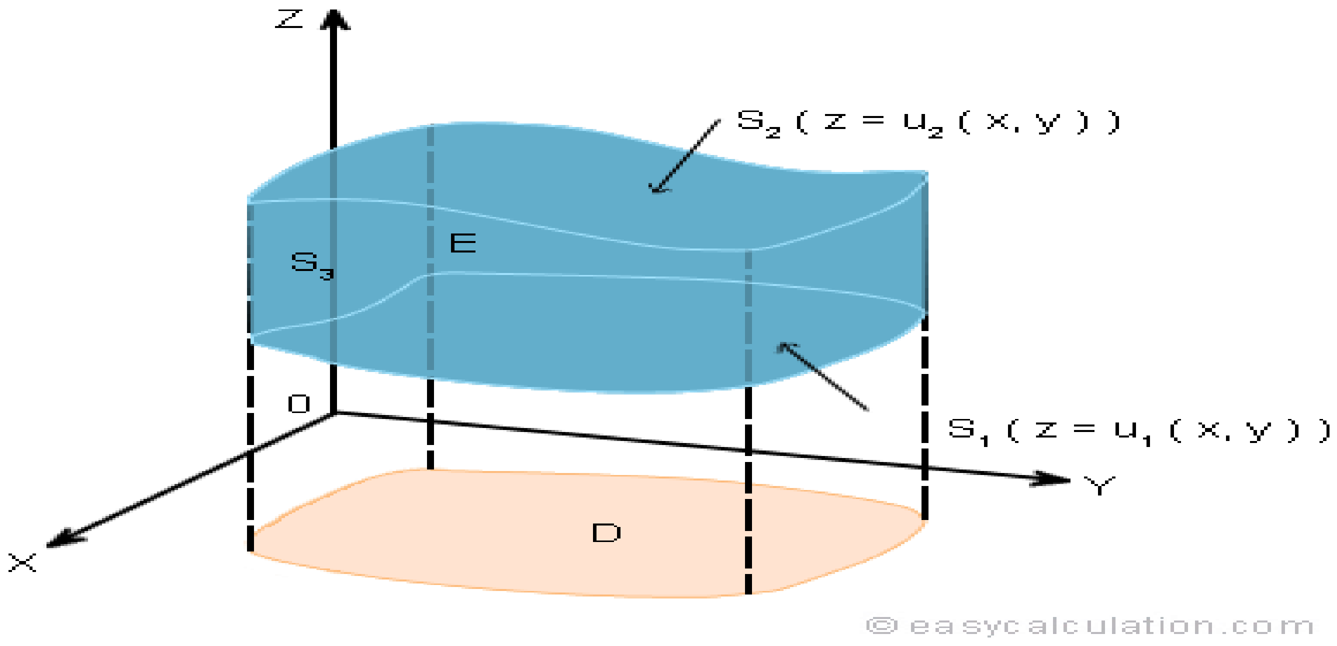
$$\oint_C \vec{V}.\overrightarrow{dr} = \int_S \int (Curl\,\vec{V}).\hat{n}dA = \int_S \int (\vec{\nabla} \times \vec{V}).\hat{n}dA$$



Gauss's Theorem

Let D be a closed and bounded region in 3-dimensional space whose boundary is a piecewise smooth surface S oriented outwards. Let V be a vector which is continuous and have continuous first order partial derivatives. Let n is a unit normal vector drawn outward to the surface S. Then:

$$\iint_S (\vec{V}.\hat{n})dA = \iiint_D (div\,\vec{V})dv = \iiint_D (\vec{\nabla}.\vec{V})dv$$



MCQ Practice Questions

In next section we will be discussing some MCQ questions from previous years question papers.

Q1. If $\vec{v} = (2xyz + 3x^2)\hat{\imath} + (1 + x^2 z)\hat{\jmath} + (x^2 y)\hat{k}$ and C is the boundary of the triangle with vertices $(1,0,0), (0,2,0), (0,0,3)$ then integral $\int_C \vec{V}.\overrightarrow{dr}$ is equal to"

(A) -1 **(B)** 1

(C) 2 **(D)** 0

Sol. By Stokes' theorem: $\oint_C \vec{V}.\overrightarrow{dr} = \int_S \int (Curl\,\vec{V}).\hat{n}dA$

$$\text{curl}(\vec{v}) = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xyz + 3x^2) & (1 + x^2 z) & (x^2 y) \end{vmatrix}$$

$$= \hat{\imath}(x^2 - x^2) - \hat{\jmath}(2xy - 2xy) + \hat{k}(2xz - 2xz) = 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k} = \vec{0}$$

$$\oint_C \vec{V}.\overrightarrow{dr} = \int_S \int (Curl\,\vec{V}).\hat{n}dA = 0$$

Hence, option **(D)** is correct answer.

Q2. If C is any simple smooth and closed curve and $f, g, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}$ are continuous in a region R bounded by closed curve C then the value of the integral $\oint_C f dx + g dy$ is:

(A) $\iint_R \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right) dxdy$

(B) $\iint_R \left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right) dxdy$

(C) $\iint_R \left(\frac{\partial g}{\partial y} - \frac{\partial f}{\partial x}\right) dxdy$

(D) $\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$

Sol. By Greens' theorem: $\oint_C f(x, y)dx + g(x, y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$

Hence, option **(D)** is correct answer.

Q3. If C is any simple smooth and closed curve and $f, g, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}$ are continuous in a region R bounded by closed curve C then which of the following is **Not** representing total are of region R:

(A) $\oint_C \; xdy$

(B) $-\oint_C \; ydx$

(C) $\oint_C \; (xdy + ydx)$

(D) $\iint_R \; dxdy$

Sol. By Greens' theorem, area of region R is given by:

Area of region R = $\oint_C \; xdy = -\oint_C \; ydx = \frac{1}{2}\oint_C \; (xdy - ydx)$

Also by double integrals: Area of region R= $\iint_R \; dxdy$

Hence, option **(C)** is correct answer.

Q4. The value of the integral $\oint_C (y^2 - x^2 + y)dx + (2xy + x)dy$ where C is the closed triangle with vertices $(0,0), (2,0), (2,4)$, taken in order is:

(A) 44 **(B)** 0

(C) 33 **(D)** 11

Sol. Comparing it with: $\oint_C fdx + gdy$

$f = (y^2 - x^2 + y)$implies $\frac{\partial f}{\partial y} = 2y + 1$, $g = (2xy + x)$ implies $\frac{\partial g}{\partial x} = 2y + 1$

By Greens' theorem:

$$\oint_C fdx + gdy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

$$= \iint_R \left((2y + 1) - (2y + 1)\right) dxdy = 0$$

Hence, option **(B)** is correct answer.

Q5. If S is the boundary of closed and bounded region D and V is the volume of this region D and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, then the value of the surface integral $\iint_S (\vec{r}.\hat{n})dA$ is:

(A) $3V$ **(B)** $6V$

(C) $2V$ **(D)** V

Sol. By Gauss's theorem: $\iint_S (\vec{r}.\hat{n})dA = \iiint_D (div\, \vec{r})dv$

Here, div$(\vec{v})$ $= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 1 + 1 + 1 = 3$

So, $\iint_S (\vec{r}.\hat{n})dA = \iiint_D 3\, dv = 3 \iiint_D dv = 3(Volume\ of\ region\ D) = 3V$

Hence, option **(A)** is correct answer.

Q6. If $\int_P^Q \vec{F}.\overrightarrow{dr}$ is independent of path of integration if:

(A) **c**url $\left(\vec{F}\right) = 0$

(B) div $(\vec{F}) = 0$

(C) grad $(\left|\vec{F}\right|) = 0$

(D) None of these

Sol. By Stokes' theorem: $\oint_P^Q \vec{F}.\overrightarrow{dr} = \int_S \int \left(Curl\ \vec{F}\right).\hat{n}dA$

So, the integral will be independent of path of integration if $\int_P^Q \vec{F}.\overrightarrow{dr} = 0$

This is possible if and only if **c**url $\left(\vec{F}\right) = 0$

Hence, option **(A)** is correct answer.

Q7. The value of the integral $\oint_C \, ds$ where C is circle of radius 2 units is:

(A) 6π **(B)** 8π

(C) 4π **(D)** 3π

Sol. The integral $\oint_C \, ds$ is actually equal to length of arc of curve C.

For a complete closed circle, the length of arc is equal to circumference of circle.

$$\oint_C \, ds = \text{Circumference of circle} = 2\pi r = 2\pi(2) = 4\pi$$

Hence, option **(C)** is correct answer.

Q8. The value of the integral $\oint_C f(x,y)dy$ where $f(x,y) = x$ and the curve C is $x = 2\cos t, y = 2\sin t, 0 \le t \le 2\pi$ is :

(A) 6π **(B)** 8π

(C) 4π **(D)** 3π

Sol. $f(x,y) = x \quad \Rightarrow f(t) = 2\cos t$

$y = 2\sin t \quad \Rightarrow \frac{dy}{dt} = 2\cos t$

$$\oint_C f(x,y)dy = \int_{t=0}^{t=2\pi}\left[f(t)\frac{dy}{dt}\right]dt = \int_{t=0}^{t=2\pi}[(2\cos t)2\cos t\,]dt$$

$$= 4\int_{t=0}^{t=2\pi} cos^2 t\, dt = 4\int_{t=0}^{t=2\pi}\left(\frac{1+\cos 2t}{2}\right)dt = 2\left[t + \frac{\sin 2t}{2}\right]_{t=0}^{t=2\pi}$$

$$= 2\left[2\pi + \frac{\sin 4\pi}{2} - 0\right] = 2[2\pi - 0] = 4\pi$$

Hence, option **(C)** is correct answer.

Q9. The value of the integral $\oint_C (xy^2 - 2y)dx + (x^2y + 3)dy$ where C is the rectangle with vertices $(-1,0), (1,0), (1,1), (-1,1)$ taken in order is:

(A) 24 **(B)** 4

(C) 10 **(D)** 0

Sol. Comparing it with: $\oint_C fdx + gdy$

$f = (xy^2 - 2y) \Rightarrow \frac{\partial f}{\partial y} = 2xy - 2,$ $\quad g = (x^2y + 3) \Rightarrow \frac{\partial g}{\partial x} = 2xy$

By Greens' theorem:

$\oint_C fdx + gdy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = \iint_R \left((2xy) - (2xy - 2)\right) dxdy$

$= 2\iint_R dxdy = 2$ (Area of rectangle with length 2 units and breadth 1 unit)

$= 2(l \times b) = 2(2 \times 1) = 4$

Hence, option **(B)** is correct answer.

Q10. The value of the integral $\oint_C (3xy^2 - 2)dx + (3x^2y + 3)dy$ where C is the simple closed curve, is:

(A) 2 **(B)** 4

(C) 1 **(D)** 0

Sol. Comparing it with: $\oint_C fdx + gdy$

$f = (3xy^2 - 2) \Rightarrow \frac{\partial f}{\partial y} = 6xy,\qquad g = (3x^2y + 3) \Rightarrow \frac{\partial g}{\partial x} = 6xy$

By Greens' theorem:

$\oint_C fdx + gdy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = \iint_R (6xy - 6xy)dxdy = 0$

Hence, option **(D)** is correct answer.

Q11. The value of the integral $\oint_C (xy^2 + y)dx + (x^2y)dy$ where C is the closed curve $x^2 + y^2 = 4$

(A) 2π

(B) 4π

(C) -4π

(D) 0

Sol. Comparing it with: $\oint_C fdx + gdy$

$f = (xy^2 + y) \Rightarrow \frac{\partial f}{\partial y} = 2xy + 1,\qquad g = (x^2y) \Rightarrow \frac{\partial g}{\partial x} = 2xy$

By Greens' theorem:

$\oint_C fdx + gdy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = \iint_R \left((2xy) - (2xy + 1)\right) dxdy$

$= -\iint_R dxdy =$ (Area of circle $x^2 + y^2 = 4$)

$= -(\pi r^2) = -(\pi(2)^2) = -4\pi$

Hence, option **(C)** is correct answer.


Thanks!